$\lim_{x\to 0}\frac{\sin x}{x}=1$, $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$, $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$

$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$, $\lim_{x\to 0}\frac{e^x-1}{x}=1$, $\lim_{x\to 0}\frac{(1+x)^\alpha-1}{\alpha x}=1$

一、极限计算

1. 等价无穷小代换

$\sin x\sim x$, $\tan x\sim x$, $\arcsin x\sim x$, $\arctan x\sim x$, $\ln(1+x)\sim x$, $e^x-1\sim x$, $(1+x)^\alpha-1\sim\alpha x$, $1-\cos x\sim\frac{1}{2}x^2$

2. Stolz 定理

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}$

3. Taylor 展开

$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$

$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$

$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$

$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

$(1+x)^\alpha=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\cdots$

$\arcsin x=x+\frac{x^3}{6}+\frac{3x^5}{40}+\cdots$

$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$

4. 洛必达法则

$\lim\frac{f(x)}{g(x)}=\lim\frac{f'(x)}{g'(x)}$

$\lim\frac{f(x)}{g(x)}=\lim\frac{f''(x)}{g''(x)}$

$\lim\frac{f(x)}{g(x)}=\lim\frac{f'''(x)}{g'''(x)}$

二、导数

1. 切线与法线方程

切线 $y-y_0=f'(x_0)(x-x_0)$

法线 $y-y_0=-\frac{1}{f'(x_0)}(x-x_0)$

2. 高阶导数

$(\sin x)^{(n)}=\sin\left(x+\frac{n\pi}{2}\right)$

$(\cos x)^{(n)}=\cos\left(x+\frac{n\pi}{2}\right)$

$(\ln x)^{(n)}=(-1)^{n-1}\frac{(n-1)!}{x^n}$

$(e^x)^{(n)}=e^x$

$(a^x)^{(n)}=a^x\ln^n a$

$(x^\alpha)^{(n)}=\alpha(\alpha-1)\cdots(\alpha-n+1)x^{\alpha-n}$

$(uv)^{(n)}=\sum_{k=0}^{n}C_n^k u^{(k)}v^{(n-k)}$

三、微分

1. 全微分

$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$

$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$

$dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv$

$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz+\frac{\partial u}{\partial t}dt$

2. 隐函数求导

$F(x,y)=0$ 则 $\frac{dy}{dx}=-\frac{F'_x(x,y)}{F'_y(x,y)}$

$F(x,y,z)=0$ 则 $\frac{\partial z}{\partial x}=-\frac{F'_x(x,y,z)}{F'_z(x,y,z)}$, $\frac{\partial z}{\partial y}=-\frac{F'_y(x,y,z)}{F'_z(x,y,z)}$

$F(x,y,z)=0$, $G(x,y,z)=0$ 则 $\frac{dy}{dx}=\frac{\begin{vmatrix}F'_z & F'_x\\ G'_z & G'_x\end{vmatrix}}{\begin{vmatrix}F'_y & F'_z\\ G'_y & G'_z\end{vmatrix}}$, $\frac{dz}{dx}=\frac{\begin{vmatrix}F'_x & F'_y\\ G'_x & G'_y\end{vmatrix}}{\begin{vmatrix}F'_y & F'_z\\ G'_y & G'_z\end{vmatrix}}$

3. 参数方程求导

$\begin{cases}x=\varphi(t)\\ y=\psi(t)\end{cases}$ 则 $\frac{dy}{dx}=\frac{\psi'(t)}{\varphi'(t)}$, $\frac{d^2y}{dx^2}=\frac{\psi''(t)\varphi'(t)-\psi'(t)\varphi''(t)}{\varphi'^3(t)}$

四、积分

1. 不定积分

$\int\frac{dx}{x^2+a^2}=\frac{1}{a}\arctan\frac{x}{a}+C$

$\int\frac{dx}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}+C$

$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$

$\int\frac{dx}{\sqrt{x^2\pm a^2}}=\ln\left|x+\sqrt{x^2\pm a^2}\right|+C$

$\int\sec x\,dx=\ln|\sec x+\tan x|+C$

$\int\sec^2 x\,dx=\tan x+C$

$\int\csc x\,dx=\ln|\csc x-\cot x|+C$

$\int\csc^2 x\,dx=-\cot x+C$

$\int\sec x\tan x\,dx=\sec x+C$

$\int\csc x\cot x\,dx=-\csc x+C$

$\int\frac{dx}{\sin x}=\ln|\csc x-\cot x|+C$

$\int\frac{dx}{\cos x}=\ln|\sec x+\tan x|+C$

$\int\frac{dx}{\sin^2 x}=-\cot x+C$

$\int\frac{dx}{\cos^2 x}=\tan x+C$

$\int\frac{dx}{\sin^2 x\cos^2 x}=\tan x-\cot x+C$

$\int\frac{dx}{\sin^2 x\cos^2 x}=\tan x+\cot

Exercise 10.1.8 (Integral 10)

$\int_0^1 \frac{\ln(1+x)}{1+x^2}dx$

$\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \int_0^1 \ln(1+x)\,d\arctan x$

$= \ln(1+x)\arctan x\Big|_0^1 - \int_0^1 \frac{\arctan x}{1+x}dx$

$= \frac{1}{2}\ln 2\arctan 1 - \int_0^1 \frac{\arctan x}{1+x}dx$

$= \frac{1}{2}\ln 2\cdot\frac{\pi}{4} - \int_0^1 \frac{\arctan x}{1+x}dx$

Then

$\int_0^1 \frac{\arctan x}{1+x}dx$

$= \int_0^1 \frac{\arctan x}{1+x}dx - \int_0^1 \frac{\arctan x}{1+x}dx + \

7. (1) $f\in BV[a,b]$，则 $f\in AC[a,b] \iff V_a^x f\in AC[a,b]$

(2) f 在 $[a,b]$ 递增，$\int_a^b f'\,dx = f(b)-f(a)$，则 $f\in AC[a,b]$

(3) $f\in BV[a,b]$，$\int_a^b |f'|\,dx = V_a^b f$，则 $f\in AC[a,b]$

(4) $f,g\in AC[a,b]$，则 $V_a^x(fg)\in AC[a,b]$

8. $f\in AC[a,b]$，则 $l(f)=\int_a^b\sqrt{1+(f'(x))^2}\,dx$

证明：只须指出，$l(f)\le\int_a^b\sqrt{1+(f'(x))^2}\,dx$

$\forall\varepsilon>0$，$\exists\,\Delta: a=x_0<x_1<\cdots<x_n=b$，$l_\Delta(f)+\varepsilon>l(f)$

$f'\in L^1[a,b]$，故存在 $g\in C[a,b]$，s.t. $\int|f'-g|\,dx<\frac{\varepsilon}{2n}$

令 $G(x)=f(a)+\int_a^x g\,dt$，则 $|G(x)-f(x)|<\frac{\varepsilon}{2n}$

$|l_\Delta(f)-l_\Delta(G)|<\varepsilon$，

$\left|\int_a^b\sqrt{1+f'^2}\,dx-\int_a^b\sqrt{1+g^2}\,dx\right|\le\int_a^b|f'-g|\,dx<\varepsilon$

故 $l_\Delta(f)<l_\Delta(G)+\varepsilon\le\int_a^b\sqrt{1+g^2}\,dx+\varepsilon<\int_a^b\sqrt{1+f'^2}\,dx+2\varepsilon$

$\Rightarrow l(f)<\int_a^b\sqrt{1+f'^2}\,dx+3\varepsilon$

9. $f_n:[0,1]\to\mathbb{R}$ 可测，以下等价

(1) $\exists\, f_{n_k}\to 0$ a.e.

(2) $\exists\,\{c_n\}\subset\mathbb{R}$，$\underline{\lim}|c_n|>0$，$\sum c_n f_n(t)$ a.e 收敛

(3) $\exists\,\{c_n\}\subset\mathbb{R}$，$\sum|c_n|=\infty$，$\sum|c_n f_n(t)|$ a.e 收敛

证明：(1) $\Rightarrow$ (2), (3) 不妨 $f_n\to 0$, a.e. $\forall k$，$\exists F_k$，$m([0,1]\setminus F_k)<\frac{1}{k}$

$\exists n_k$，$\forall n>n_k$，$|f_n(t)|<\frac{1}{2^k}$ on F_k.

$c_n=\begin{cases}1, & \text{若}\exists k,\ n=n_k\\ 0, & \text{otherwise}\end{cases}$ 即可

(2) $\Rightarrow$ (1) 显然

(3) $\Rightarrow$ (1) $G(t)=\sum|c_n f_n(t)|$，$A_k=\{t\mid G(t)\le k\}$，

$\sum|c_n|\int_{A_k}|f_n|=\int_{A_k}G<\infty \Rightarrow \underline{\lim}\int_{A_k}|f_n|=0$

$\exists n_k$，$\int_{A_k}|f_{n_k}|<\frac{1}{2^k}\Rightarrow\sum_k\int_{A_m}|f_{n_k}|<\infty$

$\Rightarrow$ a.e. $t\in A_m$，$\sum|f_{n_k}(t)|<\infty\Rightarrow f_{n_k}(t)\to 0$ a.e. $t\in A_m$

$\Rightarrow f_k(t)\to 0$ a.e.